B1 Solutions

(i) We need to identify the TF and can do so in several steps.

Step 1: There are 2 asymptotes at $\pm 90°$ so there are 2 more poles than there are zeros.

There must be 2 poles originating at $-5 \pm 2j$

so $(s+5+2j)(s+5-2j) = s^2 + 10s + (5+2j)(5-2j)$
$= \underline{s^2 + 10s + 29}$

The asymptotes approach $\rho = -2.5$ and there must be 4 poles and 2 zeros since there are 4 branches in the root locus plot.

$$\therefore \quad \rho = \frac{(-10+p_1+p_2) - (z_1+z_2)}{4-2} = \frac{p_1+p_2-10-(z_1+z_2)}{2} = -2.5$$

Step 2: Only $[-1,-4]$ is part of root loci. This means that there are either 2 poles or 2 zeros at the origin since $[-1, 0]$ must be $\underline{\text{to the left}}$ of an $\underline{\text{even}}$ number of OL poles/zeros

There are 2 poles or zeros at -1 and -4

Plugging these into ρ

if $p_1 = -1 \quad p_2 = -4$
then $z_1 = z_2 = 0$

$$\frac{-1-4-10-(0+0)}{2} = \frac{-15}{2} \neq -2.5$$

if $z_1 = -1 \quad p_2 = 0$
$z_2 = -4 \quad p_1 = 0$

$$\frac{-10-(-1-4)}{2} = \frac{-5}{2} = -2.5 \quad \checkmark$$

Step 3: Point of arrival near -2 since OL poles move from origin to intersect real axis here.

$$\therefore \quad \boxed{G_p(s) = \frac{(s+1)(s+4)}{s^2(s^2+10s+29)}}$$

(ii) This is a Type-2 system, so $\boxed{e_{ss}=0}$

Alternatively,

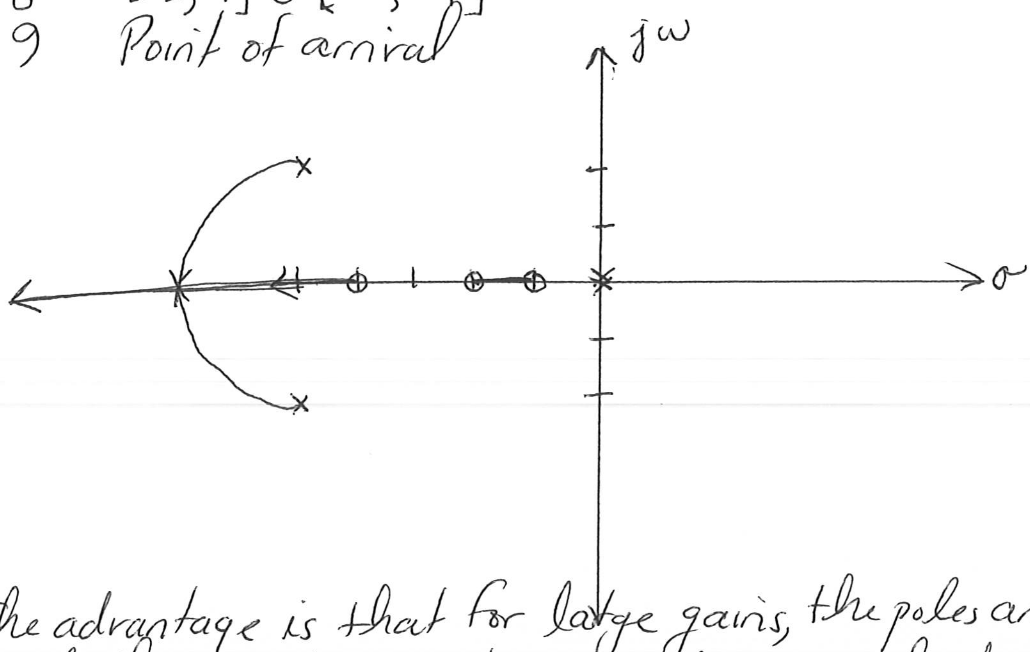
$$e_{ss} = \lim_{s\to 0} \frac{s\,R(s)}{1+G(s)} = \lim_{s\to 0} \frac{s\left(\frac{1}{s^2}\right)}{1+\frac{(s+1)(s+4)}{s^2(s^2+10s+29)}}$$

$$= \lim_{s\to 0} \frac{\left(\frac{1}{s}\right)s^2(s^2+10s+29)}{s^2(s^2+10s+29)+(s+1)(s+4)} = \lim_{s\to 0} \frac{s(s^2+10s+29)}{s^2(s^2+10s+29)+(s+1)(s+4)} = \frac{0}{4} = \boxed{0}$$

(iii) The previous question shows that the steady-state error is zero for a ramp. Since there is already a double pole at the origin, adding a third one won't really improve performance.

(iv) $G_c = s+2$ so $G_pG_c = \frac{(s+1)(s+2)(s+4)}{s^2(s^2+10s+29)}$

Rule 4 4 O.L. poles - 3 O.L. zeros = 1 asymptote
Rule 5 asymptote is 180°
Rule 6 N/A
Rule 8 $[-2,-1] \cup [-\infty,-4]$
Rule 9 Point of arrival



(v) The advantage is that for large gains, the poles are purely real thus stability is better and without high-frequency oscillations.

B2 Solutions

$$\dot{x} = -x + z - x^2 z - y^2$$
$$\dot{y} = z$$
$$\dot{z} = -4(x+1)^2 - z + u$$

(i) Equilibrium points mean that $\dot{x} = \dot{y} = \dot{z} = 0$

therefore $\dot{y} = \boxed{0 = z}$

$$\dot{z} = -4(x+1)^2 - 0 + u = 0$$
$$(x+1)^2 = 0 \qquad \boxed{x = -1}$$
$$\dot{x} = 0 = -1 + 0 - (-1)^2 \cdot 0 - y^2$$
$$y^2 = 1 \qquad \boxed{y = \pm 1}$$

(ii) Linearise around $(-1, 1, 0)$

$$x = x_0 + \epsilon x_1 = -1 + \epsilon x_1 \qquad \dot{x} = \epsilon \dot{x}_1$$
$$y = y_0 + \epsilon y_1 = 1 + \epsilon y_1 \qquad \dot{y} = \epsilon \dot{y}_1$$
$$z = z_0 + \epsilon z_1 = \epsilon z_1 \qquad \dot{z} = \epsilon \dot{z}_1$$

y is easy as it's already linear

$$\epsilon \dot{y}_1 = \epsilon z_1 \Rightarrow \boxed{\dot{y}_1 = z_1}$$

let's consider x next

$$\epsilon \dot{x}_1 = -(-1 + \epsilon x_1) + \epsilon z_1 - (-1 + \epsilon x_1)^2 (\epsilon z_1) - (1 + \epsilon y_1)^2$$
$$\epsilon \dot{x}_1 = \cancel{1} - \epsilon x_1 + \epsilon z_1 - (1 - 2\epsilon x_1 + \epsilon^2 x_1^2)(\epsilon z_1) - (\cancel{1} + 2\epsilon y_1 + \epsilon^2 y_1^2)$$
$$\epsilon \dot{x}_1 = -\epsilon x_1 + \cancel{\epsilon z_1} - \cancel{\epsilon z_1} + \cancel{2\epsilon^2 x_1 z_1} - \cancel{\epsilon^3 x_1^2 z_1} - 2\epsilon y_1 - \cancel{\epsilon^2 y_1^2}$$
$$\therefore \boxed{\dot{x}_1 = -x_1 - 2y_1}$$

z turns out to be easy

$$\epsilon \dot{z}_1 = -4(-1 + \epsilon x_1 + 1)^2 - \epsilon z_1$$
$$\epsilon \dot{z}_1 = -\cancel{4\epsilon^2 x_1^2} - \epsilon z_1 \qquad \boxed{\dot{z}_1 = -z_1}$$

(iii) This is straightforward

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

Note that the A matrix is upper triangular so eigenvalues are diagonal entries and the system is marginally stable.

(iv) Need to find rank of $\mathcal{C} = [B \;\; AB \;\; A^2B]$

$$AB = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \text{so } \mathcal{C} = \underline{\begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}}$$

$$A^2B = A(AB) = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

The columns of $\mathcal{C}$ are linearly independent so the matrix is full-rank and the system is controllable.

(v) We only need to modify the 3rd pole. Assuming $K = [0 \;\; 0 \;\; k_3]$

$$A - BK = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & k_3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1-k_3 \end{pmatrix}$$

we want $\lambda_3 = -3 \qquad \lambda_3 = -1 - k_3$ so $k_3 = 2$

therefore $\boxed{K = [0 \;\; 0 \;\; 2]}$

(vi) The system would be overdetermined so no as choice of K is not unique.

B3 Solutions

(i) This is difficult to do w/ great accuracy but the key is the phase plot. We initially begin with -360° and ramp up quickly to -180° at $\omega = 1$. There is also a resonance peak here so this indicates a quadratic lag that is <u>unstable</u>

so $\dfrac{1}{s^2 - 2\zeta\omega_n s + \omega_n^2}$ where $\omega_n \simeq 1$ since there is not much damping approximate damping ratio as $\zeta = 0.1$

$\therefore$ TF must contain $\dfrac{1}{s^2 - 0.2s + 1}$

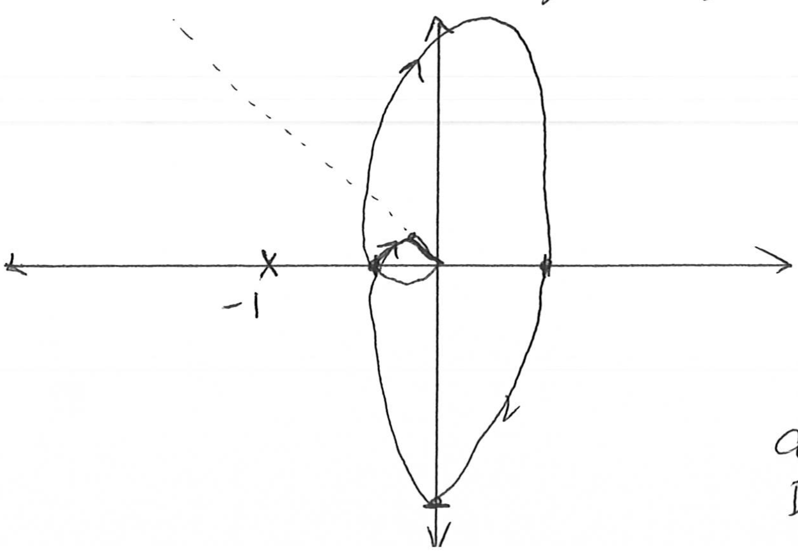
There is an increase in phase from 10^0 to 10^2 indicating a zero around $\omega = 10$. This is confirmed by the less negative slope of the magnitude plot.

There is a decrease in phase from 10^2 to 10^3 indicating a pole around $\omega = 100$.

The overall TF is approximately

$$G_p = \frac{(s+10)}{(s^2 - 0.2s + 1)(s+100)}$$

(ii)



The start is -360° at roughly 0dB

at -270°, amplitude or magnitude reaches the peak value

at -180°, magnitude <u>below</u> 0dB

approaches -135°

Decreases to $-\infty$dB at -180°

For the present system, there are no encirclements of -1 and 2 unstable poles so there would be 2 unstable poles for the closed-loop system.

If K_p were increased, eventually -1 would be encircled twice thus stabilising the system.

(iii) There are several options such as a PI controller or a phase-lag.

let's go w/ a phase-lag $G_c = \frac{(s+z)}{(s+p)}$ $z > p$

(iv) Resulting Bode diagram is: