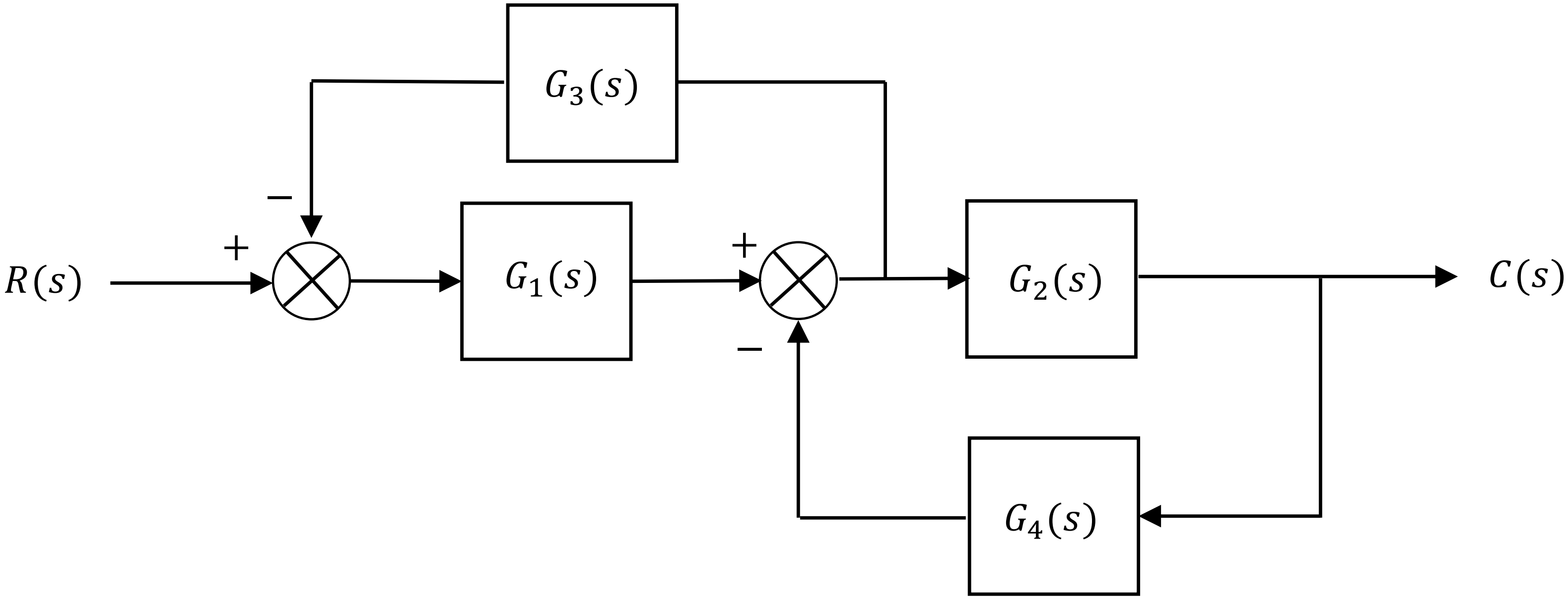
Q(A i): block diagram 6 marks

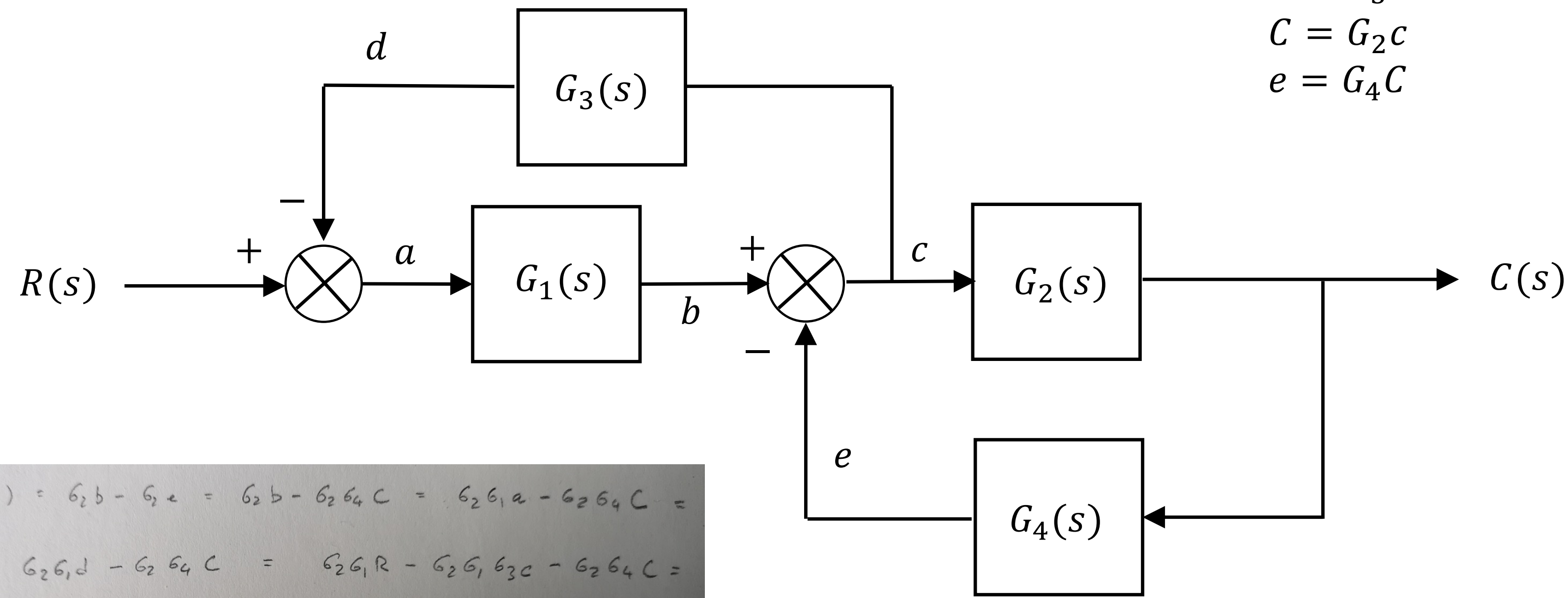


Correct result: 6 Marks

Mistakes in the block diagram manipulation: 3-4 (3 if correct moves across pickoff or sum junction points

Some procedure: 2 Marks

$$a = R - d$$
$$b = G_1 a$$
$$c = b - e$$
$$d = G_3 c$$
$$C = G_2 c$$
$$e = G_4 C$$



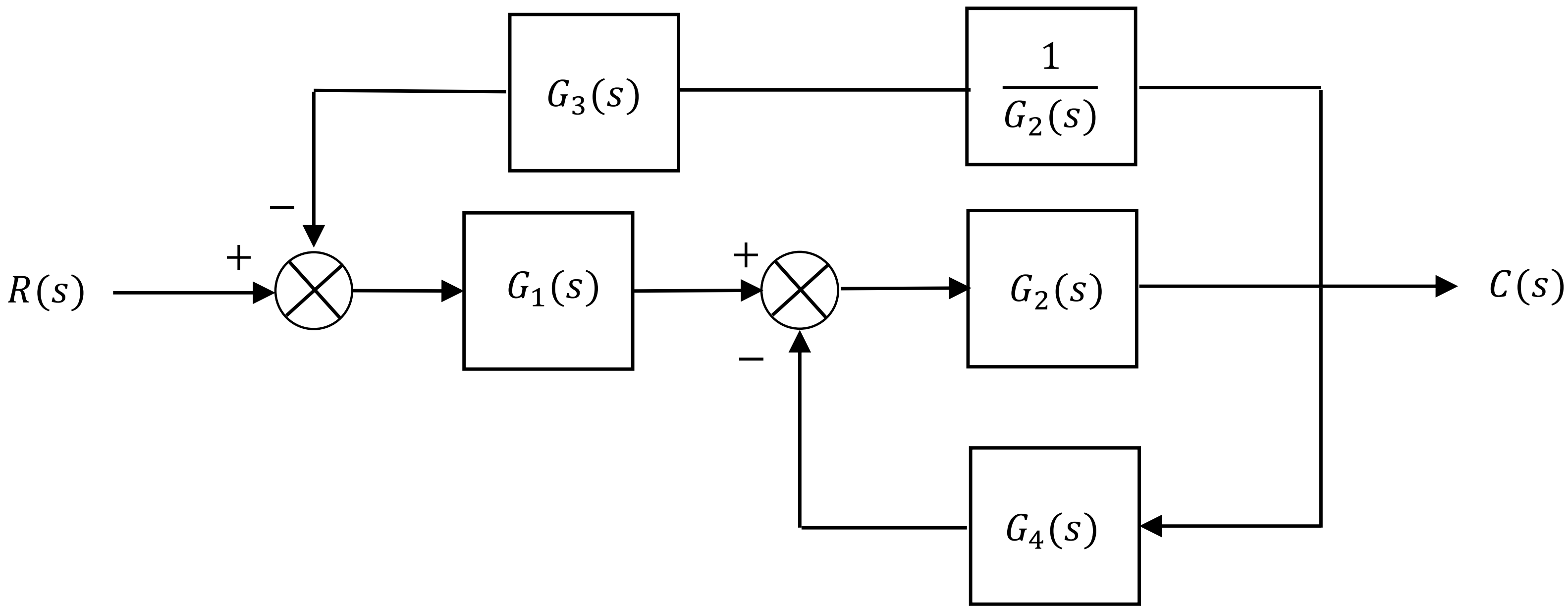
$$C = G_2 c = G_2(b-e) = G_2 b - G_2 e = G_2 b - G_2 G_4 C = G_2 G_1 a - G_2 G_4 C =$$

$$G_2 G_1 R - G_2 G_1 d - G_2 G_4 C = G_2 G_1 R - G_2 G_1 G_3 c - G_2 G_4 C =$$

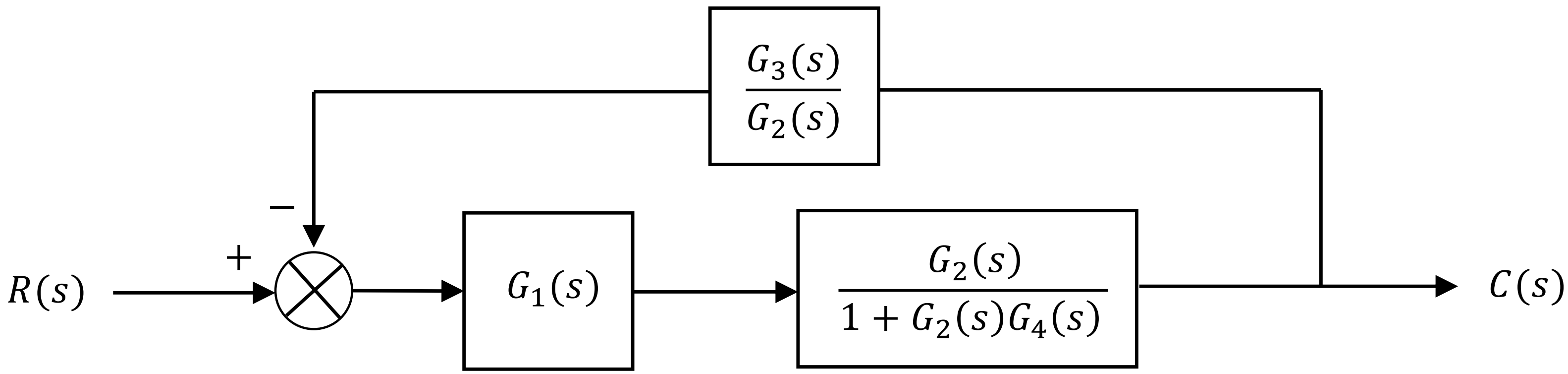
$$G_2 G_1 R - \cancel{G_2} G_1 G_3 \frac{C}{\cancel{G_2}} - G_2 G_4 C = G_1 G_2 R - G_1 G_3 C - G_2 G_4 C$$

$$C + G_1 G_3 C + G_2 G_4 C = G_1 G_2 R$$

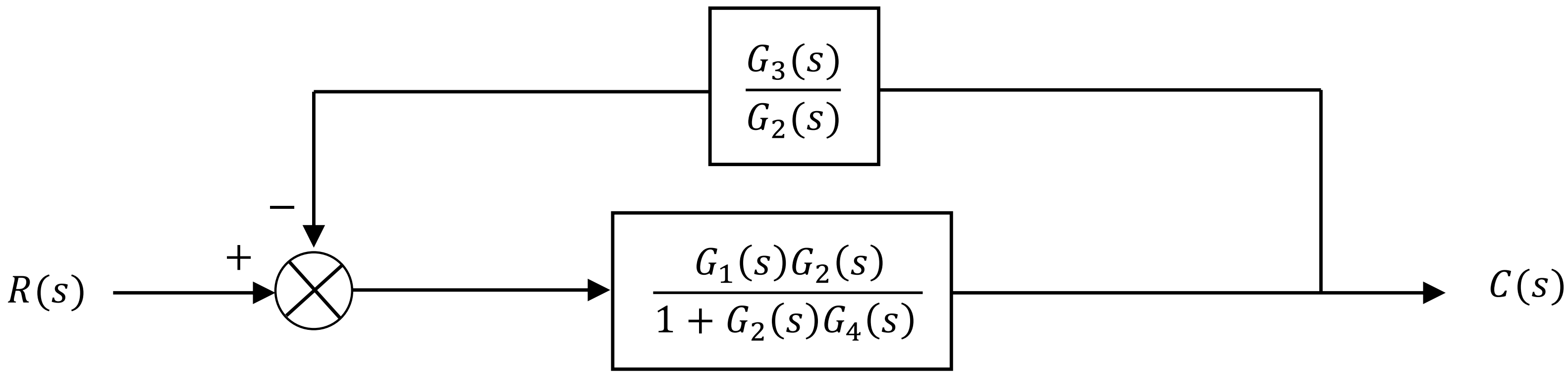
$$\frac{C}{R} = \frac{G_1 G_2}{1 + G_1 G_3 + G_2 G_4}$$



R(s)
+
−
G3(s)
1
G2(s)
G1(s)
+
−
G2(s)
C(s)
G4(s)



$\frac{G_3(s)}{G_2(s)}$
−
+
$R(s)$
$G_1(s)$
$\frac{G_2(s)}{1 + G_2(s)G_4(s)}$
$C(s)$



$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1(s)G_2(s)}{1+G_2(s)G_4(s)}}{1+\dfrac{G_1(s)G_2(s)}{1+G_2(s)G_4(s)}\dfrac{G_3(s)}{G_2(s)}} = \frac{G_1(s)G_2(s)}{1+G_2(s)G_4(s)+G_1(s)G_3(s)}$$

Q(A ii): find transfer function 4 Marks

The PO is given by expression: $PO = 100 \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ which allows to evaluate damping factor $\zeta = 0.1$.

1 Mark

The 5% settling time reads: $T_s = \frac{3}{\zeta\omega_n}$ which after evaluating ζ from PO allows to evaluate $\omega_n = 10$ rad/s.

1 Mark

Correct result: 1 Mark for ζ + 1 Mark for ω_n

Q(A iii) PID: 3 marks

The differential equation for the PID controller can be stated as $m(t) = K_p e(t) + K_i \int_0^t e(t)dt + K_d \frac{de(t)}{dt}$ **[1]**

There are many different ways of writing the transfer function of a PID controller. The general form is

$$G_c(s) = K_c + \frac{K_i}{s} + K_d s$$ **[2]**

where K_c is the proportional gain (often K_p is used), K_i is the integral gain used to reduce steady state errors and K_d is the derivative gain used to respond to the rate of error change.

[3 mark]

Q(A iv): To find the transfer function corresponding to the time response: 5 Marks

Definition of TF 1 mark

$$3 - 5e^{-t} + 2e^{-2t}$$

First find the Laplace transform of the response and arrange:

$$(3 - 5e^{-t} + 2e^{-2t})\,\theta(t) \rightarrow$$

2 Marks

$$\frac{3}{s} - \frac{5}{s+1} + \frac{2}{s+2} =$$

$$\frac{s+6}{s(s+1)(s+2)}$$

Then since the Laplace transform of a unit step is $1/s$ we obtain the transfer function:

$$G(s) = \frac{s+6}{(s+1)(s+2)}$$

2 Marks

Correct result +1;
LT of input +1

Q(A v): Linearization:

7 Marks

a) $$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 4x^2 = e^{-t}$$

eq:

$x_0 = \frac{1}{2}$, $t_0 = 0$ $\qquad$ $x = x_0 + \Delta x$, $t = t_0 + \Delta t$ $\qquad$ $4x_0 = e^{-t_0}$

$$\frac{d^2\Delta x}{dt^2} + \frac{d\Delta x}{dt} + 4(x_0^2 + 2x_0\Delta x + \Delta x^2) = e^{-t_0}e^{-\Delta t}$$

$$\frac{d^2\Delta x}{dt^2} + \frac{d\Delta x}{dt} + 4x_0^2 + 8x_0\Delta x = e^{-t_0}[1 - \Delta t]$$

$$\frac{d^2\Delta x}{dt^2} + \frac{d\Delta x}{dt} + 4x_0^2 + 8x_0\Delta x = e^{-t_0} - e^{-t_0}\Delta t$$

=

$t_0 = 0$

$x_0 = 1/2$

$$\frac{d^2\Delta x}{dt^2} + \frac{d\Delta x}{dt} + 4\Delta x = -\Delta t$$

b) The system is (when linearized)

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 4x = u(t)$$ where the driving is $u(t) = -t$

LT:

$$s^2X(s) + sX(s) + 4X(s) = U(s)$$

thus

TF: $$\frac{X(s)}{U(s)} = \frac{1}{s^2+s+4}$$

(a)

3 Mark (procedure)

1 Mark (correct)

(b)

2 Mark procedure

1 Mark for recognizing that u(t) = -t

Q(A vi): Consider transfer function: 5 Marks

$$G(s) = \frac{50}{s^2 + 2s + 25}$$

- Natural frequency is: $\omega_n = \sqrt{25} s^{-1} = 5\ s^{-1}$
- Damping factor: $2\zeta\omega_n = 2 \rightarrow \zeta = \frac{1}{\omega_n} = 0.2$
- Peak time: $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\sqrt{24}}$ sec ≈ 0.64 sec
- Settling time: $T_s = 4T = \frac{4}{\zeta\omega_n} = 4$ sec
- Percentage overshoot: $PO = 100 \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 100 \exp\left(-\frac{\pi}{\sqrt{24}}\right) = 52.66\%$

3 Marks

If 1 missing -1
If 2 missing -2
If all missing -3

1 Marks

Formulas matter
If not units then -1 (if some units ok)

Q(A vii): State space representation: 6 Mark

Handle nominator and denominator of the transfer function separately.

Denominator:

- Convert to ode: $\ddot{y} + 4\dot{y} + 2y = r$
- Identify state variables: $x_1 = y, x_2 = \dot{y}$ and so $\dot{x}_1 = \dot{y}, \dot{x}_2 = \ddot{y}$ 1 Mark
- So: $\dot{x}_2 + 4x_2 + 2x_1 = r$ and:

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$ 2 Mark

Nominator:

- $x_1 = x_1;\ \dot{x}_1 = x_2$ 1 Mark
- $y = x_2 + 3x_1$ 1 Mark
- $y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ 1 Mark

Q(A viii): Gain and Phase at $\omega = 1$ rad/s 4 Marks

$$G(s) = \frac{s^3 + 2s^2}{s^4 + s^3 + 1}$$

$$G(j\omega)_{\omega\to 1} = \frac{-j\omega^3 - 2\omega^2}{\omega^4 - j\omega^3 + 1} \xrightarrow{\omega=1} \frac{-j-2}{1-j+1} = -\frac{2+j}{2-j} = -\frac{3}{5} - \frac{4}{5}j$$ 2 Marks

Gain:

$$|G(j\omega)_{\omega\to 1}| = \frac{\sqrt{25}}{5} = 1$$

2 Marks

Phase:

$$\arg G(j\omega)_{\omega\to 1} = \arg\left(-\frac{3}{5} - \frac{4}{5}j\right) = \tan^{-1}\frac{4}{3} = 53.12^{\circ}$$