SESA6085 - Tutorial Workbook Solutions

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Basic Probability Theory

A1.1:
Using the binary partition version of Bayes theorem to calculate $P(D|C)$ where $P(D|C) = \frac{P(C|D)P(D)}{P(C|D)P(D)+P(C|\bar{D})P(\bar{D})}$. Giving $P(D|C) = 0.43$

A1.2:
Using Bayes theorem:

- $P(A_1|F) = 0.051$
- $P(A_2|F) = 0.070$
- $P(A_3|F) = 0.134$
- $P(A_4|F) = 0.745$

Note, if correct $\sum P(A_i|F) = 1.0$ as if there has been a failure it must have been caused by one of the A_i events.

Continuous Distributions

A2.1:
See Figure 1 below.

A2.2:
(a) $R(800) = 0.977$
(b) $R(1600) = 0.023$

A2.3:
Option 3 has the higher reliability and should be selected.

- $R_1(800) = 0.894$
- $R_2(800) = 0.977$
- $R_3(800) = 0.994$

A2.4:
The probability of a rejection is given by $P(< 18) + P(> 23)$ which is equal

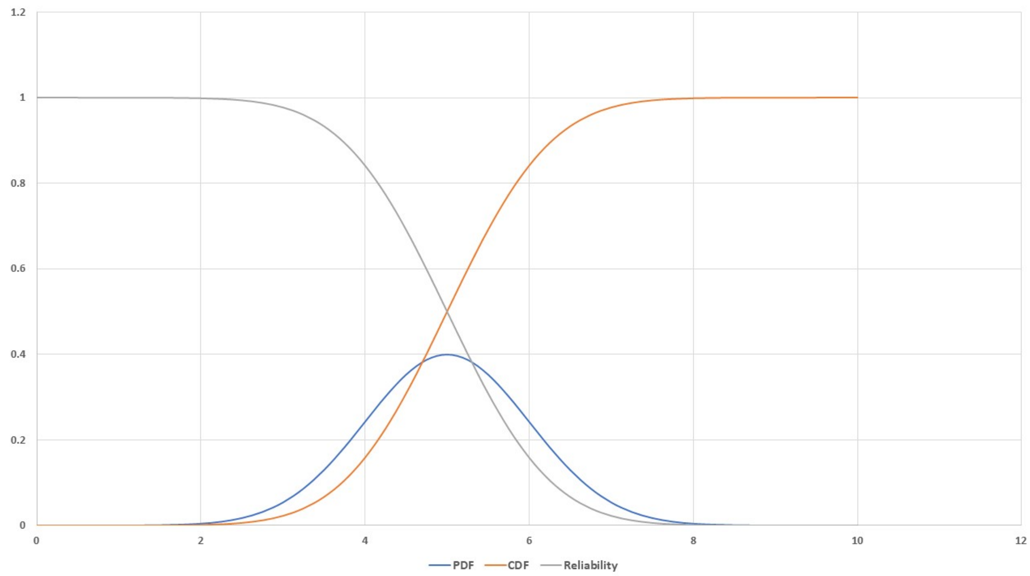

Figure 1:

to $F(18) + R(23) = 0.0846 + 0.0647 = 0.149$.

A2.5:
The aim is to determine t such that $R(t) = 0.3$. This can be determined graphically by plotting the reliability function and reading of the value of t giving $R(t) = 0.3$. Alternatively an optimisation or goal seek can be used in Excel or the inverse CDF function in Matlab or Python. $t = 13.448$ years. For this case, the equation for the inverse can also be calculated and used to solve for t. Note that most software uses a two variable Weibull distribution so the failure free time, γ needs to be accounted for.

A2.6:
Using the functions in Excel, Matlab or Python, $R(40) = 1 - F(40) = 0.905$.

Parameter Estimation

A3.1:
Using MLE, $\hat{\mu} = 2.916$ and $\hat{\sigma} = 0.593$. Using the Fisher information matrix

$\mathrm{Var}(\hat{\mu}) = 0.00390$ and $\mathrm{Var}(\hat{\sigma}) = 0.00195$ giving 95% confidence bounds of $2.79 \leq \hat{\mu} \leq 3.04$ and $0.51 \leq \hat{\sigma} \leq 0.68$. Finally, the probability of failure between 30 and 55 years is 0.174.

A3.2:
Recall the PDF and reliability functions for an exponential distribution are,

$$f(t) = \lambda e^{-\lambda t} \qquad\qquad R(t) = e^{-\lambda t}.$$

The general expression for the likelihood function then becomes,

$$L(\lambda) = \{\prod_{i=1}^{n_u} \lambda e^{-\lambda t}\}\{\prod_{i=1}^{n_{C_R}} e^{-\lambda T_R}\}$$

where n_u and n_{C_R} are the number of uncensored and right censored data points respectively and T_R is the censoring time. Solving the likelihood equations by taking partial derivatives gives,

$$\hat{\lambda} = \frac{n_u}{\sum_{i=1}^{n_u} t_i + n_{C_R} T_R} = 0.1073$$

A3.3:
In this case the general expression for the likelihood function becomes,

$$L(\lambda) = \{\prod_{i=1}^{n_u} \lambda e^{-\lambda t}\}\{\prod_{i=1}^{n_{C_R}} e^{-\lambda T_R}\}\{\prod_{i=1}^{n_{C_L}} 1 - e^{-\lambda T_L}\},$$

where n_u, n_{C_R} and n_{C_L} are the number of uncensored, right and left censored data points respectively. T_R and T_L are the right and left censoring times. $\hat{\lambda} = 0.4148$.

A3.4:
This is a competing risk model. First the PDF function must be defined and then the parameters for both PDFs determined via MLE. The optimisation problem can be solved numerically using Excel, for example. The parameters should be $\mu_1 = 17.972$, $\sigma_1 = 6.184$, $\mu_2 = 23.248$ and $\sigma_2 = 0.766$.

Multi-variate Distributions

A4.1:

$$\boldsymbol{\mu} = \begin{bmatrix} 1.0775 \\ 1.8067 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0.0854 & 0.0106 \\ 0.0106 & 0.1532 \end{bmatrix}$$

A4.2:

The probability of scrap is defined as,

$$P_{\text{scrap}} = F(7.5, \infty) + F(\infty, 1.4) - F(7, 5, 1.4) = 0.0474,$$

where F defines the CDF of a multi-variate normal distribution which can be calculated using Matlab or Python. The probability of rework is therefore,

$$P_{\text{rework}} = 1 - F(8.5, 2.4) - (F(7.5, \infty) - F(7.5, 2.4)) - (F(\infty, 1.4) - F(8.5, 1.4))$$

$$P_{\text{rework}} = 0.0801$$

Reliability Modelling

A5.1:
See the solutions in Figure 2.

A5.2:
Applying block diagram decomposition the system can be simplified so that R_E is either present or absent. The resulting overall system reliability is $R = 0.9848$.

A5.3:
The table of sub-system and overall car reliabilities is presented in Table 1. The tyres drive the reliability of the system, note how the reliability of the

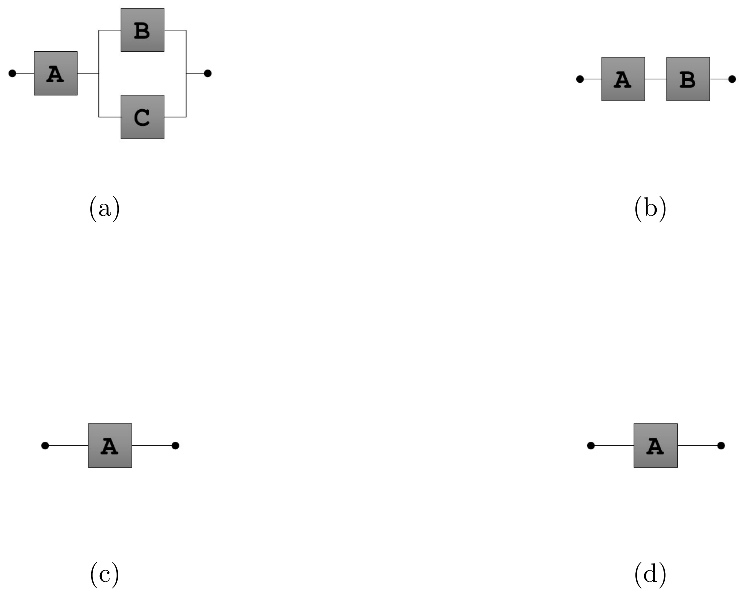

Figure 2: Reliability block diagrams for simplification.

tyre sub-system closely matches that of the car. This is the sub-system which should be targetted for design improvement. Also note the numerical error in calculating very small probabilities in the tire m/n sub-system resulting in negative values.

Table 1: Overall microcar and sub-system reliabilities.

t	R_{Engine}	$R_{steering}$	R_{Brakes}	$R_{Transmission}$	R_{Tyres}	R_{Seats}	R_{Cargo}	R_{car}
1000	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00
1500	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00
2000	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00
2500	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00
3000	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00
3500	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00
4000	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.000e+00
4500	1.000e+00	1.000e+00	1.000e+00	1.000e+00	9.978e-01	1.000e+00	1.000e+00	9.978e-01
5000	1.000e+00	1.000e+00	1.000e+00	1.000e+00	6.691e-01	1.000e+00	1.000e+00	6.691e-01
5500	1.000e+00	1.000e+00	1.000e+00	1.000e+00	6.463e-02	1.000e+00	1.000e+00	6.463e-02
6000	1.000e+00	1.000e+00	1.000e+00	1.000e+00	3.160e-04	9.863e-01	1.000e+00	3.116e-04
6500	1.000e+00	1.000e+00	1.000e+00	1.000e+00	4.864e-08	8.602e-01	9.816e-01	4.107e-08
7000	1.000e+00	1.000e+00	1.000e+00	1.000e+00	1.600e-13	5.903e-01	9.238e-01	8.726e-14
7500	1.000e+00	1.000e+00	1.000e+00	1.000e+00	-1.069e-16	3.064e-01	8.345e-01	-2.734e-17
8000	1.000e+00	9.987e-01	9.976e-01	1.000e+00	1.425e-16	1.208e-01	7.349e-01	1.260e-17
8500	1.000e+00	9.830e-01	9.617e-01	1.000e+00	-4.419e-17	3.679e-02	6.369e-01	-9.789e-19
9000	9.996e-01	9.259e-01	8.208e-01	1.000e+00	-1.851e-16	8.821e-03	5.451e-01	-6.762e-19
9500	9.961e-01	8.001e-01	5.356e-01	9.964e-01	-3.766e-18	1.692e-03	4.618e-01	-1.252e-21
10000	9.854e-01	6.009e-01	2.178e-01	9.603e-01	-5.494e-23	2.628e-04	3.882e-01	-6.942e-28

A5.4:
$R_{\text{series}} = 0.159$, $R_{\text{parallel}} = 0.292$ and $R_{\text{standby}} = 0.891$. The standby equation can be solved using Python, Matlab or Excel using a simple trapezoidal integration or a numerical integration routine in Python or Matlab.

A5.5:
$R = 0.8383$ where the system's reliability has been calculated under the assumption of a load sharing system.

A5.6:
The most reliable system costs £78.50 with $R = 0.967$. The cheapest system costs £34.00 with $R = 0.587$. The most reliable system for a maximum of £60 is, $R = 0.948$.

Robust Design

A6.1:
The minimum is at $x_1 = -3.6893$ and $x_2 = 13.6300$ giving a minimum value of $f(\boldsymbol{x}) = -16.6440$. This can be solved relatively easily using any global optimisation algorithm such as Excel's solver tool with the evolutionary option turned on.

A6.2:
The optimum is at $x_1 = 1.9879$ and $x_2 = 4.4871$ where the mean of $f(\boldsymbol{x})$ subject to the variations in x_1 and x_2 is approximately 34.79. This can be solved relatively easily by using a Monte Carlo analysis to determine the mean of $f(\boldsymbol{x})$ as x_1 and x_2 is varied. The Monte Carlo analysis requires the definition of a set of two normally distributed random numbers whose means shift with x_1 and x_2. These can then be used to calculate $f(\boldsymbol{x})$ and therefore the mean. A optimisation algorithm can then vary x_1 and x_2 to minimise this value. Note that the answer will be dependent on the number of Monte Carlo evaluations performed.

Optimal Maintenance Scheduling

A7.1:
Calculating both of these costs depends heavily on the step size used in the calculation of $M(t)$ and $tf(t)$ integral. $c(t_p) = 11.40$ and $c(t_{pa}) = 11.70$ when the step size equals 100 hours. $c(t_p) = 12.02$ and $c(t_{pa}) = 11.66$ when the step size equals 1 hour.

A7.2:
Again the optimal time is dependent on the step size used to calculate $M(t)$ but should be approximately 8500 hours when a step size of 500 hours is used. This will converge to approximately 8300 hours with a step size of 25 hours.

Project Uncertainty Management

A8.1:
The contract offer from company 3 should be accepted based on the lower expected cost of $50,000 and the potential for $\approx 70\%$ of maintenance cases to be cheaper.

A8.2:
There is an 80% probability of a delay. The expected delay is 3 days with an expected cost of £160,000.

A8.3:
The expected cost without crashing is approximately $11,500, while crashing results in an expected cost of approximately $2,300.